Chap. 4 Systems of ODEs. Phase Plane. Qualitative Methods

The methods discussed in this chapter use elementary linear algebra (Sec. 4.0). First we present another method for solving higher order ODEs that is different from the methods of Chap. 3. This method consists of converting any $n$th-order ODE into a system of $n$ first-order ODEs, and then solving the system obtained (Secs. 4.1–4.4 for homogeneous linear systems). We also discuss a totally new way of looking at systems of ODEs, that is, a qualitative approach. Here we want to know the behavior of families of solutions of ODEs without actually solving these ODEs. This is an attractive method for nonlinear systems that are difficult to solve and can be approximated by linear systems by removing nonlinear terms. This is called linearization (Sec. 4.5). In the last section we solve nonlinear systems by the method of undetermined coefficients, a method you have seen before in Secs. 2.7 and 2.10.

Sec. 4.0 For Reference: Basics of Matrices and Vectors

This section reviews the basics of linear algebra. Take a careful look at Example 1 on p. 130. For this chapter you have to know how to calculate the characteristic equation of a square $2 \times 2$ (or at most a $3 \times 3$) matrix and how to determine its eigenvalues and eigenvectors. To obtain the determinant of $A - \lambda I$, denoted by det$(A - \lambda I)$, you first have to compute $A - \lambda I$. Note that $\lambda$ is a scalar, that is, a number. In the following calculation, the second equality holds because of scalar multiplication and the third equality by matrix addition:

$$
A - \lambda I = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -4.0 - \lambda & 4.0 - \lambda \\ -1.6 - \lambda & 1.2 - \lambda \end{bmatrix}.
$$

Then you compute (see solution to Prob. 5 of Problem Set 3.1 on how to calculate determinants)

$$
det(A - \lambda I) = \begin{vmatrix} -4.0 - \lambda & 4.0 \\ -1.6 & 1.2 - \lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = (\lambda + 2)(\lambda + 0.8) = 0.
$$

The roots of the characteristic polynomial are called the eigenvalues of matrix $A$. Here they are $\lambda_1 = -2$ and $\lambda_2 = 0.8$. The calculations for the eigenvector corresponding to $\lambda_1 = -2$ are shown in the book. To determine an eigenvector corresponding to $\lambda_2 = 0.8$, you first have to substitute $\lambda = 0.8$ into the system

$$
(A - \lambda I)x = \begin{bmatrix} -4.0 - \lambda & 4.0 \\ -1.6 & 1.2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (-4.0 - \lambda)x_1 + 4.0x_2 \\ -1.6x_1 + (-1.2 - \lambda)x_2 \end{bmatrix} = 0.
$$

This gives

$$
\begin{bmatrix} (-4.0 - 0.8)x_1 + 4.0x_2 \\ -1.6x_1 + (-1.2 - 0.8)x_2 \end{bmatrix} = \begin{bmatrix} -4.8x_1 + 4.0x_2 \\ -1.6x_1 + 2.0x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

Using the second equation, that is, $-1.6x_1 + 2.0x_2 = 0$, and setting $x_1 = 1.0$, gives $2.0x_2 = 0$, so that $x_2 = 1.6/2.0 = 0.8$. Thus an eigenvector corresponding to the eigenvalue $\lambda_2 = 0.8$ is $x^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$. Also note that eigenvectors are only determined up to a nonzero constant. We could have chosen $x_1 = 3.0$ and gotten $x_2 = 4.8/2.0 = 2.4$, thereby obtaining an eigenvector $\begin{bmatrix} 3 \\ 2.4 \end{bmatrix}$ corresponding to $\lambda_2$. 
13. Conversion of a single ODE to a system. This conversion is an important process, which always follows the pattern shown in formulas (9) and (10) of Sec. 4.1. The present equation \( y'' + 2y' - 24y = 0 \) can be readily solved as follows. Its characteristic equation (directly from Sec. 2.2) is \( \lambda^2 + 2\lambda - 24 = (\lambda - 4)(\lambda + 6) \). It has roots 4, -6 so that the general solution of the ODE is \( y = c_1 e^{4t} + c_2 e^{-6t} \). The point of the problem is to explain the relation between systems of ODEs and single ODEs and their solutions. To explore this new idea, we chose a simple problem, whose solution can be readily obtained. (Thus we are not trying to explain a more complicated method for a simple problem!) In the present case the formulas (9) and (10) give \( y_1, y_2 \) and

\[
\begin{align*}
  y_1' &= y_2 \\
  y_2' &= 24y_1 - 2y_2
\end{align*}
\]
(because the given equation can be written $y'' = 24y - 2y'$, hence $y''_1 = 24y_1 - 2y_2$, but $y''_1 = y'_2$).

In matrix form (as in Example 3 of the text) this is

$$y' = Ay = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$ 

Then you compute

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 24 & -2 \end{bmatrix} - \begin{bmatrix} -\lambda & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 24 & -2 - \lambda \end{bmatrix}.$$ 

Then the characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix} = $$

$$\lambda^2 + 2\lambda - 24 = (\lambda - 4)(\lambda + 6) = 0.$$ 

The eigenvalues, which are the roots of the characteristic equation, are $\lambda_1 = 4$ and $\lambda_2 = -6$. For $\lambda_1$, you obtain an eigenvector from (13) in Sec. 4.0 with $\lambda = \lambda_1$, that is,

$$(A - \lambda_1 I)x = \begin{bmatrix} -4 & 1 \\ 24 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 + x_2 \\ 24x_1 - 6x_2 \end{bmatrix} = 0.$$ 

From the first equation $-4x_1 + x_2 = 0$ you have $x_2 = 4x_1$. An eigenvector is determined only up to a nonzero constant. Hence, in the present case, a convenient choice is $x_1 = 1$, which, when substituted into the first equation, gives $x_2 = 4$. Thus an eigenvector corresponding to $\lambda_1 = 4$ is

$$x^{(1)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$ 

(The second equation gives the same result and is not needed.) For the second eigenvalue, $\lambda_2 = -6$, you proceed the same way, that is,

$$(A - \lambda_2 I)x = \begin{bmatrix} 6 & 1 \\ 24 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + x_2 \\ 24x_1 + 4x_2 \end{bmatrix} = 0.$$ 

You now have $6x_1 + x_2 = 0$, hence $x_2 = -6x_1$, and can choose $x_1 = 1$, thus obtaining $x_2 = -6$. Thus an eigenvector corresponding to $\lambda_2 = -6$ is

$$x^{(2)} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$ 

Expressing the two eigenvectors in transpose (T) notation, you have

$$x^{(1)} = [1, 4]^T \quad \text{and} \quad x^{(2)} = [1, -6]^T.$$ 

Multiplying these by $e^{4t}$ and $e^{-6t}$, respectively, and taking a linear combination involving two arbitrary constants $c_1$ and $c_2$ gives a general solution of the present system in the form

$$y = c_1[1, 4]^T e^{4t} + c_2[1, -6]^T e^{-6t}.$$
In components, this is, corresponding to the answer on p. A9
\[ y_1 = c_1 e^{4t} + c_2 e^{-6t} \]
\[ y_2 = 4c_1 e^{4t} - 6c_2 e^{-6t}. \]

Here you see that \( y_1 = y \) is a general solution of the given ODE, and \( y_2 = y' = y' \) is the derivative of this solution, as had to be expected because of the definition of \( y_2 \) at the beginning of the process.

Note that you can use \( y_2 = y_1' \) for checking your result.

Sec. 4.2 Basic Theory of Systems of ODEs. Wronskian

The ideas are similar to those of Secs. 1.7 and 2.6. You should know what a Wronskian is and how to compute it. The theory has no surprises and you will use it naturally as you do your homework exercises.

Sec. 4.3 Constant-Coefficient Systems. Phase Plane Method

In this section we study the phase portrait and show five types of critical points. They are improper nodes (Example 1, pp. 141–142, Fig. 82), proper nodes (Example 2, Fig. 83, p. 143), saddle points (Example 3, pp. 143–144, Fig. 84), centers (Example 4, p. 144, Fig. 85), and spiral points (Example 5, pp. 144–145, Fig. 86). There is also the possibility of a degenerate node as explained in Example 6 and shown in Fig. 87 on p. 146.

Example 2. Details are as follows. The characteristic equation is
\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = (\lambda - 1)^2 = 0.
\]

Thus \( \lambda = 1 \) is an eigenvalue. Any nonzero vector with two components is an eigenvector because \( Ax = x \) for any \( x \); indeed, \( A \) is the \( 2 \times 2 \) unit matrix! Hence you can take \( x^{(1)} = [1 \quad 0]^T \) and \( x^{(2)} = [0 \quad 1]^T \) or any other two linearly independent vectors with two components. This gives the solution on p. 143.

Example 3. \((1 - \lambda)(-1 - \lambda) = (\lambda - 1)(\lambda + 1) = 0\), and so on.

Problem Set 4.3. Page 147

1. General solution. The matrix of the system \( y'_1 = y_1 + y_2, y'_2 = 3y_1 - y_2 \) is
\[
A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}.
\]

From this you have the characteristic equation
\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 3 & -1 - \lambda \end{vmatrix} = (1 - \lambda)(-1 - \lambda) - 3 = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0.
\]

You see that the eigenvalues are \( \pm 2 \). You obtain eigenvectors for \(-2\) from \( 3x_1 + x_2 = 0 \), say \( x_1 = 1 \), \( x_2 = -3 \) (recalling that eigenvectors are determined only up to an arbitrary nonzero factor). Thus
for $\lambda = -2$ you have an eigenvector of $[1 \quad -3]^T$. Similarly, for $\lambda = 2$ you obtain an eigenvector from $-x_1 + x_2 = 0$, say, $[1 \quad 1]^T$. You thus obtain the general solution
\[
y = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.
\]

On p. A10 this is written in terms of components.

7. General solution. Complex eigenvalues. Write down the matrix
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]

Then calculate
\[
A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & 0 \\ -1 - \lambda & 1 \\ 0 & -1 - \lambda \end{bmatrix}.
\]

From this you obtain the characteristic equation by taking the determinant of $A - \lambda I$, that is,
\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 \\ -1 - \lambda & 1 \\ 0 & -1 - \lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ -1 & 1 \end{vmatrix} = -\lambda(-\lambda - 1) = -\lambda^3 - \lambda - \lambda = -\lambda^3 + 2\lambda
\]
\[
= -\lambda(\lambda^2 + 2) = -\lambda(\lambda - \sqrt{2}i)(\lambda + \sqrt{2}i) = 0.
\]

The roots of the characteristic equation are $0, \pm \sqrt{2}i$. Thus the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -\sqrt{2}i$, and $\lambda_3 = +\sqrt{2}i$.

For $\lambda_1 = 0$ you obtain an eigenvector from
\[
\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

This gives us a system consisting of 3 homogeneous linear equations, that is
\[
-x_1 + x_3 = 0, \\
-x_2 = 0, \\
\text{so that } x_3 = x_1.
\]

Thus, if we choose $x_1 = 1$, then $x_3 = 1$. Also $x_2 = 0$ from the first equation. Thus $[1 \quad 0 \quad 1]^T$ is an eigenvector for $\lambda_1 = 0$. 
For $\lambda_2 = -\sqrt{2}i$, we obtain an eigenvector as follows:

$$
\begin{bmatrix}
\sqrt{2}i & 1 & 0 \\
-1 & \sqrt{2}i & 1 \\
0 & -1 & \sqrt{2}i
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
$$

This gives the following system of linear equations:

$$
\sqrt{2}ix_1 + x_2 = 0 \quad \text{so that} \quad x_2 = -\sqrt{2}ix_1.
$$

$$
-x_1 + \sqrt{2}x_2 + x_3 = 0,
$$

$$
-x_2 + \sqrt{2}ix_3 = 0.
$$

Substituting $x_3 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation gives us

$$
-x_1 + \sqrt{2}x_2 + x_3 = -x_1 + (\sqrt{2}i)(-\sqrt{2}i)x_1 + x_3
= -x_1 + 2x_1 + x_3 = x_1 + x_3 = 0 \quad \text{hence} \quad x_1 = -x_3.
$$

[Note that, to simplify the coefficient of the $x_1$-term, we used that $(\sqrt{2}i)(-\sqrt{2}i) = -(\sqrt{2})(\sqrt{2}) \\
(\sqrt{-1})(\sqrt{-1}) = -(2)(-1) = -2$, where $i = \sqrt{-1}$.] Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = -\sqrt{2}i$. Thus the eigenvector for $\lambda_2 = -\sqrt{2}i$ is

$$
\begin{bmatrix}
x_1 & x_2 & x_3
\end{bmatrix}^T = [1 \quad -\sqrt{2}i \quad -1]^T.
$$

For $\lambda_3 = \sqrt{2}i$, we obtain the following system of linear equations:

$$
-\sqrt{2}ix_1 + x_2 = 0 \quad \text{so that} \quad x_2 = \sqrt{2}ix_1.
$$

$$
-x_1 - \sqrt{2}ix_2 + x_3 = 0,
$$

$$
-x_2 - \sqrt{2}ix_3 = 0 \quad \text{so that} \quad x_2 = -\sqrt{2}ix_1.
$$

Substituting $x_2 = \sqrt{2}ix_1$ (obtained from the first equation) into the second equation

$$
x_1 = -\sqrt{2}x_2 + x_3 = -\sqrt{2}i\sqrt{2}ix_1 + x_3 = 2x_1 + x_3, \quad \text{hence} \quad x_1 = -x_3.
$$

(Another way to see this is to note that, $x_2 = \sqrt{2}ix_1$ and $x_2 = -\sqrt{2}ix_3$, so that $\sqrt{2}ix_1 = -\sqrt{2}ix_3$ and hence $x_1 = -x_3$.) Setting $x_1 = 1$ gives $x_3 = -1$, and $x_2 = \sqrt{2}i$. Thus the eigenvector for $\lambda_3 = \sqrt{2}i$ is $[1 \quad \sqrt{2}i \quad 1]^T$, as was to be expected from before. For more complicated calculations, you might want to use Gaussian elimination (to be discussed in Sec. 7.3).

Together, we obtain the general solution

$$
y = c_1^* \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{0t} + c_2^* \begin{bmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it} + c_3^* \begin{bmatrix} 1 \\ \sqrt{2}i \\ -1 \end{bmatrix} e^{-\sqrt{2}it}
$$
15. Initial value problem. Solving an initial value problem for a system of ODEs is similar to that of solving an initial value problem for a single ODE. Namely, you first have to find a general solution and then determine the arbitrary constants in that solution from the given initial conditions.
To solve Prob. 15, that is,
\[ y_1' = 3y_1 + 2y_2, \]
\[ y_2' = 2y_1 + 3y_2, \]
\[ y_1(0) = 0.5, \quad y_2(0) = 0.5, \]
write down the matrix of the system
\[ A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}. \]

Then
\[ A - \lambda I = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{bmatrix} \]
and determine the eigenvalues and eigenvectors as before. Solve the characteristic equation
\[ \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{vmatrix} = (3 - \lambda)(3 - \lambda) - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0. \]
You see that the eigenvalues are \( \lambda = 1 \) and \( \lambda = 5 \). For \( \lambda = 1 \) obtain an eigenvector from \((3 - 1)x_1 + 2x_2 = 0\), say, \( x_1 = 1, x_2 = -2 \). Similarly, for \( \lambda = 5 \) obtain an eigenvector from \((3 - 5)x_1 + 2x_2 = 0\), say, \( x_1 = 2, x_2 = 1 \). You thus obtain the general solution
\[ y = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}. \]

From this, and the initial conditions, you have, setting \( t = 0 \),
\[ y(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}. \]

From the second component you obtain \(-c_1 + c_2 = -0.5\), hence \( c_2 = -0.5 + c_1 \). From this, and the first component, you obtain
\[ c_1 + c_2 = c_1 - 0.5 + c_1 = 0.5, \quad \text{hence} \ c_1 = 0.5. \]

Conclude that \( c_2 = -0.5 + c_1 = -0.5 + 0.5 = 0 \) and get, as on p. A10,
\[ y = 0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t = \begin{bmatrix} 0.5e^t \\ -0.5e^t \end{bmatrix}. \]

Written in components, this is
\[ y_1 = 0.5e^t, \]
\[ y_2 = -0.5e^t. \]
Sec. 4.4 Criteria for Critical Points. Stability

The type of critical point is determined by quantities closely related to the eigenvalues of the matrix of the system, namely, the trace \( p \), which is the sum of the eigenvalues, the determinant \( q \), which is the product of the eigenvalues, and the discriminant \( \Delta \), which equals \( p^2 - 4q \); see (5) on p. 148. Whereas, in Sec. 4.3, we used the phase portrait to graph critical points, here we use algebraic criteria to identify critical points. Table 4.1 (p. 149) is important in identification. Table 4.2 (p. 150) gives different types of stability. You will use both tables in the problem set.

Problem Set 4.4. Page 151

7. Saddle point. We are given the system

\[
\begin{align*}
y'_1 &= y_1 + 2y_2, \\
y'_2 &= 2y_1 + y_2.
\end{align*}
\]

From the matrix of the system

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.
\]

From (5) of Sec. 4.4, you get \( p = 1 + 1 = 2, \ q = 1^2 - 2^2 = -3, \) and \( \Delta = p - 4q = 2^2 - 4(-3) = 4 + 12 = 16. \) Since \( q < 0, \) we have a saddle point at \((0, 0)\). Indeed, the characteristic equation

\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0
\]

has the real roots \(-1, 3, \) which have opposite signs, as it should be according to Table 4.1 on p. 149. Also, \( q < 0 \) implies that the critical point is unstable. Indeed, saddle points are always unstable.

To find a general solution, determine eigenvectors. For \( \lambda = -1 \) you find an eigenvector from \((1 - \lambda)x_1 + 2x_2 = 2x_1 + 2x_2 = 0, \) say, \( x_1 = 1, \ x_2 = -1, \) giving \( [1, -1]^T. \) Similarly, for \( \lambda = 3 \) you have \(-2x_1 + 2x_2 = 0, \) say, \( x_1 = 1, \ x_2 = 1, \) so that an eigenvector is \( [1, 1]^T. \) You thus obtain the general solution

\[
y = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}
\]

and in components is

\[
\begin{align*}
y_1 &= c_1 e^{-t} + c_2 e^{3t}, \\
y_2 &= -c_1 e^{-t} + c_2 e^{3t}.
\end{align*}
\]

11. Damped oscillations. The ODE \( y'' + 2y' + 2y = 0 \) has the characteristic equation

\[
\lambda^2 + 2\lambda + 2 = (\lambda + 1 + i)(\lambda + 1 - i) = 0.
\]

You thus obtain the roots \(-1 - i \) and \(-1 + i \) and the corresponding real general solution (Case III, complex conjugate, Sec. 2.2)

\[
y = e^{-t}(A \cos t + B \sin t)
\]

(see p. A10). This represents a damped oscillation.
Convert this to a system of ODEs
\[ \begin{align*}
y_1' &= y_2 \\
y_2'' &= y_1'' = y_2 = -2y_1 - 2y_2.
\end{align*} \]

Write this in matrix form,
\[ \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \]

Hence
\[ \begin{align*}
p &= \lambda_1 + \lambda_2 = (-1 - i) + (-1 + i) = -2, \\
q &= \lambda_1 \lambda_2 = (-1 - i)(-1 + i) = 1 - i^2 = 2, \\
\Delta &= (\lambda_1 - \lambda_2)^2 = (-2i)^2 = -4.
\end{align*} \]

Since \( p \neq 0 \) and \( \Delta < 0 \), we have spirals by Table 4.1(d). Furthermore, these spirals are stable and attractive by Table 4.2(a).

Since the physical system has damping, energy is taken from it all the time, so that the motion eventually comes to rest at \((0, 0)\).

17. **Perturbation.** If the entries of the matrix of a system of ODEs are measured or recorded values, errors of measurement can change the type of the critical point and thus the entire behavior of the system.

The unperturbed system
\[ \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y} \]

has a center by Table 4.1(c). Now Table 4.1(d) shows that a slight change of \( p \) (which is 0 for the undisturbed system) will lead to a spiral point as long as \( \Delta \) remains negative.

The answer (a) on p. A10 suggests \( b = -2 \). This changes the matrix to
\[ \begin{bmatrix} -2 & -1 \\ -6 & -2 \end{bmatrix}. \]

Hence you now have \( p = -4, q = 4 - 6 = -2 \), so that you obtain a saddle point. Indeed, recall that \( q \) is the determinant of the matrix, which is the product of the eigenvalues, and if \( q \) is negative, we have two real eigenvalues of opposite signs, as is noted in Table 4.1(b).

To verify all the answers to Prob. 17 given on p. A10, calculate the quantities needed for the perturbed matrix
\[ \tilde{\mathbf{A}} = \begin{bmatrix} b & 1+b \\ -4+b & b \end{bmatrix} \]
in the form
\[ \begin{align*}
\tilde{p} &= 2b, \\
\tilde{q} &= \det \tilde{\mathbf{A}} = b^2 - (1+b)(-4+b) = 3b + 4, \\
\tilde{\Delta} &= \tilde{p} - 4\tilde{q} = 4b^2 - 12b - 16,
\end{align*} \]
and then use Tables 4.1 and 4.2.
Sec. 4.5 Qualitative Methods for Nonlinear Systems

The remarkable basic fact in this section is the following. Critical points of a nonlinear system may be investigated by investigating the critical points of a linear system, obtained by linearization of the given system—a straightforward process of removing nonlinear terms. This is most important because it may be difficult or impossible to solve such a nonlinear system or perhaps even to discuss general properties of solutions.

In the process of linearization, a critical point to be investigated is first moved to the origin of the phase plane and then the nonlinear terms of the transformed system are omitted. This results in a critical point of the same type in almost all cases—exceptions may occur, as is discussed in the text, but this is of lesser importance.

Problem Set 4.5. Page 159

5. Linearization. To determine the critical points of the given system, we set \( y_1' = 0 \) and \( y_2' = 0 \), that is,

\[
\begin{align*}
y_1' &= y_2 = 0, \\
y_2' &= -y_1 + \frac{1}{2} y_1^2 = 0.
\end{align*}
\]

If we factor the second ODE, that is,

\[
y_2' = -y_1(1 - \frac{1}{2} y_1^2) = 0,
\]

we get \( y_1 = 0 \) and \( y_2 = 2 \). This gives us two critical points of the form \((y_1, y_2)\), that is, \((0, 0)\) and \((2, 0)\). We now discuss one critical point after the other.

The first is at \((0, 0)\), so you need not move it (you do not need to apply a translation). The linearized system is simply obtained by omitting the nonlinear term \( \frac{1}{2} y_2^2 \). The linearized system is

\[
\begin{align*}
y_1' &= y_2, \\
y_2' &= -y_1
\end{align*}
\]

in vector form

\[
y' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y.
\]

The characteristic equation is

\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0
\]

so that \( \lambda_1 = i, \lambda_2 = -i \). From this we obtain

\[
\begin{align*}
p &= \lambda_1 + \lambda_2 = -i + i = 0, \\
q &= \lambda_1 \lambda_2 = (i)(-i) = 1, \\
\Delta &= (\lambda_1 - \lambda_2)^2 = (-2i)^2 = -4.
\end{align*}
\]

Since \( p = 0, q = 1 \) and we have pure imaginary eigenvalues, we conclude, by Table 4.1(c) in Sec. 4.4, that we have a center at \((0, 0)\).

Turn to \((2, 0)\). Make a translation such that \((y_1, y_2) = (2, 0)\) becomes \((\tilde{y}_1, \tilde{y}_2) = (0, 0)\). Notation: Note that the tilde over the variables and eigenvalues denotes the transformed variables. Nothing needs to be done about \( y_2 \), so we set \( \tilde{y}_2 = \tilde{y}_2 \). For \( y_1 = 2 \) we must have \( \tilde{y}_1 = 0 \); thus set \( y_1 = 2 + \tilde{y}_1 \). This step of translation is always the same in our further work. And if we must translate
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\((y_1, y_2) = (a, b)\), we set \(y_1 = a + \tilde{y}_1, y_2 = b + \tilde{y}_2\). The two translations give separate equations, so there is no difficulty.

Now transform the system. The derivatives always simply give \(y'_1 = \tilde{y}'_1, y'_2 = \tilde{y}'_2\). We thus obtain

\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= -y_1 + \frac{1}{2}y_1^2
\end{align*}
\]

(by factorization)

\[
\begin{align*}
y'_1 &= (\tilde{2} - \tilde{y}_1)(1 - \frac{1}{2}(2 + \tilde{y})) \\
y'_2 &= -y_1 - \frac{1}{2}\tilde{y}_1^2 \\
\end{align*}
\]

(by substitution)

Thus we have to consider the system (with the second equation obtained by the last two equalities in the above calculation)

\[
\begin{align*}
\tilde{y}'_1 &= \tilde{y}_2, \\
\tilde{y}'_2 &= \tilde{y}_1 - \frac{1}{2}\tilde{y}_1^2.
\end{align*}
\]

Hence the system, linearized at the critical \((2, 0)\), is obtained by dropping the term \(-\frac{1}{2}\tilde{y}_1^2\), namely,

\[
\tilde{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{y}.
\]

From this we determine the characteristic equation

\[
\det(\tilde{A} - \tilde{\lambda} I) = \begin{vmatrix} -\tilde{\lambda} & -1 \\ 1 & -\tilde{\lambda} \end{vmatrix} = \tilde{\lambda}^2 - 1 = (\tilde{\lambda} + 1)(\tilde{\lambda} - 1) = 0.
\]

It has eigenvalues \(\tilde{\lambda} = -1\) and \(\tilde{\lambda} = 1\). From this we obtain

\[
\begin{align*}
\tilde{\rho} &= \tilde{\lambda}_1 + \tilde{\lambda}_2 = -1 + 1 = 0, \\
\tilde{q} &= \tilde{\lambda}_1\tilde{\lambda}_2 = (-1)(1) = -1, \\
\tilde{\Delta} &= (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 = (-1 - 1)^2 = 4.
\end{align*}
\]

Since \(\tilde{q} < 0\) and the eigenvalues are real with opposite sign, we have a saddle point by Table 4.1(b) in Sec. 4.4, which is unstable by Table 4.2(c).

9. **Converting nonlinear ODE to a system. Linearization. Critical points.** Transform \(y'' - 9y + y^3 = 0\) into a system by the usual method (see Theorem 1, p. 135) of setting

\[
\begin{align*}
y_1 &= y, \\
y_2 &= y', \quad \text{so that} \quad y'_1 = y' = y_2, \quad \text{and} \\
y'_2 &= y'' = 9y - y^3 = 9y_1 - y_1^3.
\end{align*}
\]
Thus the nonlinear ODE, converted to a system of ODEs, is
\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= 9y_1 - y_3.
\end{align*}
\]
To determine the local critical points, we set the right-hand sides of the ODEs in the system of ODEs to 0, that is, \(y'_1 = y_2 = 0, y'_2 = 0\). From this, and the second equation, we get
\[
y'_2 = 9y_1 - y_3 = y_1(9 - y_1^2) = y_1(3 + y_1)(3 - y_1) = 0.
\]
The critical points are of the form \((y_1, y_2)\). Here we see that \(y_2 = 0\) and \(y_1 = 0, -3, +3\), so that the critical points are \((0, 0), (-3, 0), (3, 0)\).

Linearize the system of ODEs at \((0, 0)\) by dropping the nonlinear term, obtaining
\[
\begin{align*}
y'_1 &= y_2, \\
y'_2 &= 9y_1
\end{align*}
\]
in vector form
\[
y' = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} y.
\]
From this compute the characteristic polynomial, noting that,
\[
\begin{align*}
\lambda^2 - 9 &= (\lambda + 3)(\lambda - 3) = 0.
\end{align*}
\]
The eigenvalues \(\lambda_1 = -3, \lambda_2 = 3\). From this we obtain
\[
\begin{align*}
p &= \lambda_1 + \lambda_2 = -3 + 3 = 0, \\
q &= \lambda_1\lambda_2 = (-3)(3) = -9, \\
\Delta &= (\lambda_1 - \lambda_2)^2 = (-3 - 3)^2 = (-6)^2 = 36.
\end{align*}
\]
Since \(q < 0\) and the eigenvalues are real with opposite signs, we conclude that \((0, 0)\) is a saddle point (by Table 4.1) and, as such, is unstable (Table 4.2).

Turn to \((-3, 0)\). Make a translation such that \((y_1, y_2) = (-3, 0)\) becomes \((\tilde{y}_1, \tilde{y}_2) = (0, 0)\). Set \(y_1 = -3 + \tilde{y}_1, y_2 = \tilde{y}_2\). Then
\[
\begin{align*}
y'_2 &= y_1(9 - y_1^2) \\
&= (-3 + \tilde{y}_1)[9 - (-3 + \tilde{y}_1)^2] \\
&= (-3 + \tilde{y}_1)[9 - (9 - 6\tilde{y}_1 + \tilde{y}_1^2)] \\
&= (-3 + \tilde{y}_1)(6\tilde{y}_1 - \tilde{y}_1^2) \\
&= -18\tilde{y}_1 + 9\tilde{y}_1^2 - \tilde{y}_1^3 \\
&= \tilde{y}'_2.
\end{align*}
\]
Thus
\[
\tilde{y}'_2 = -18\tilde{y}_1 + 9\tilde{y}_1^2 - \tilde{y}_1^3.
\]
Also, by differentiating \(y_1\) in \(y_1 = -3 + \tilde{y}_1\), then using \(y'_1 = y_2\) from above and then using that \(y_2 = \tilde{y}_2\), you obtain
\[
y'_1 = \tilde{y}'_1 = y_2 = \tilde{y}_2.
\]
Together we have the transformed system
\[
\tilde{y}_1' = \tilde{y}_2, \\
\tilde{y}_2' = -18\tilde{y}_1 + 9\tilde{y}_2 - \tilde{y}_3.
\]

To obtain the linearized transformed system, we have to drop the nonlinear terms. They are the quadratic term \(9\tilde{y}_2\) and the cubic term \(-\tilde{y}_3\). Doing so, we obtain the system
\[
\tilde{y}_1' = \tilde{y}_2, \\
\tilde{y}_2' = -18\tilde{y}_1.
\]

Expressing it in vector form you have
\[
\tilde{y}' = \tilde{A}\tilde{y} = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}.
\]

From this we immediately compute the characteristic determinant and obtain the characteristic polynomial, that is,
\[
\det(\tilde{A} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -18 & -\lambda \end{vmatrix} = \lambda^2 + 18 = 0.
\]

We see that the eigenvalues are complex conjugate, that is, \(\tilde{\lambda}_1 = \sqrt{-18} = (\sqrt{18})(\sqrt{-1}) = (\sqrt{9\cdot 2})(i) = 3\sqrt{2}i\). Similarly, \(\tilde{\lambda}_2 = -3\sqrt{2}i\). From this, we calculate
\[
\tilde{p} = \tilde{\lambda}_1 + \tilde{\lambda}_2 = 3\sqrt{2}i + (-3\sqrt{2}i) = 0, \\
\tilde{q} = \tilde{\lambda}_1\tilde{\lambda}_2 = 18, \\
\tilde{\Delta} = (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 = (6\sqrt{2}i)^2 = -72.
\]

Looking at Tables 4.1, p. 149, and Table 4.2, p. 150, we conclude as follows. Since \(\tilde{p} = 0\), \(\tilde{q} = 18 > 0\), and \(\tilde{\lambda}_1 = 3\sqrt{2}i\), \(\tilde{\lambda}_2 = -3\sqrt{2}i\) are pure imaginary \(\text{(see below)}\), we conclude that \((-3, 0)\) is a center from part (c) of Table 4.1. From Table 4.2(b) we conclude that the critical point \((-3, 0)\) is stable, and indeed a center is stable.

**Remark on complex numbers.** Complex numbers are of the form \(a + bi\), where \(a, b\) are real numbers. Now if \(a = 0\), so that the complex number is of the form \(bi\), then this complex number is **pure imaginary** (or **purely imaginary**) \(\text{(i.e., it has no real part} \ a\). This is the case with \(3\sqrt{2}i\) and \(-3\sqrt{2}i\)! Thus \(6 + 5i\) is not pure imaginary, but \(5i\) is pure imaginary.

Similarly the third critical point \((3, 0)\) is a center. If you had trouble with this problem, you may want to do all the calculations for \((3, 0)\) without looking at our calculations for \((-3, 0)\), unless you get very stuck.

13. **Nonlinear ODE.** We are given a nonlinear ODE \(y'' + \sin y = 0\), which we transform into a system of ODEs by the usual method of Sec. 4.1 (Theorem 1) and get
\[
y_1' = y_2, \\
y_2' = -\sin y_1.
\]
Find the location of the critical points by setting the right-hand sides of the two equations in the system to 0, that is, \( y_2 = 0 \) and \( -\sin y_1 = 0 \). The sine function is zero at \( 0, \pm \pi, \pm 2\pi, \ldots \) so that the critical points are at \( (\pm n\pi, 0), n = 0, 1, 2, \ldots \).

Linearize the system of ODEs at \( (0,0) \) approximating \( \sin y_1 \approx y_1 \) (see Example 1 on p. 153, where this step is justified by a Maclaurin series expansion). This leads to the linearized system

\[
\begin{align*}
y_1' &= y_2 \\
y_2' &= -y_1
\end{align*}
\]

in vector form \( \mathbf{y}' = A\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} \).

Using (5) in Sec. 4.4 we have

\[
\begin{align*}
p &= a_{11} + a_{22} = 0 + 0 = 0, \\
q &= \det A = 0 \cdot 0 - (-1)(-1) = 1, \\
\Delta &= p^2 - 4q = 0^2 - 4 \cdot 1 = -4.
\end{align*}
\]

From this, and Table 4.1(c) in Sec. 4.4, we conclude that \( (0,0) \) is a center. As such it is always stable, as Table 4.2(b) confirms. Since the sine function has a period of \( 2\pi \), we conclude that \( (0,0), (\pm 2\pi, 0), (\pm 4\pi, 0), \ldots \) are centers.

Consider \( (\pi, 0) \). We transform the critical points to \( (0,0) \) as explained at the beginning of Sec. 4.5. This is done by the translation \( y_1 = \pi + \tilde{y}_1, y_2 = \tilde{y}_2 \). We now have to determine the transformed system:

\[
\begin{align*}
\tilde{y}_1 &= y_1 - \pi & \text{so} & & \tilde{y}_1' &= y_1' = y_2 & \text{so that} & & \tilde{y}_1' &= \tilde{y}_2 \\
\tilde{y}_2 &= y_2 & \text{so} & & \tilde{y}_2' &= y_2' = -\sin y = -\sin (\pi + \tilde{y}_1).
\end{align*}
\]

Now

\[
-\sin(\pi + \tilde{y}_1) = \sin(-\pi + \tilde{y}_1) \quad \text{(since sine is an odd function, App. 3, Sec. A3.1)}
\]

\[
= \sin(-\tilde{y}_1) = -\sin \tilde{y}_1 \cdot \cos \pi - \cos \tilde{y}_1 \cdot \sin \pi \quad \text{[by (6), Sec. A3.1 in App. 3]}
\]

\[
= \sin \tilde{y}_1 \quad \text{(since } \cos \pi = -1, \sin \pi = 0).\]

Hence

\[
\tilde{y}_1' = \tilde{y}_2, \\
\tilde{y}_2' = -\sin \tilde{y}_1.
\]

Linearization gives the system

\[
\begin{align*}
\tilde{y}_1' &= \tilde{y}_2 \\
\tilde{y}_2' &= \tilde{y}_1
\end{align*}
\]

in vector form \( \tilde{\mathbf{y}}' = \tilde{A}\tilde{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tilde{\mathbf{y}} \).

We compute

\[
\begin{align*}
\tilde{p} &= \tilde{a}_{11} + \tilde{a}_{22} = 0 + 0 = 0, \\
\tilde{q} &= \det \tilde{A} = 0 \cdot 0 - 1 \cdot 1 = -1, \\
\Delta &= \tilde{p}^2 - 4\tilde{q} = 0^2 - 4 \cdot (-1) = 4.
\end{align*}
\]
Since $\tilde{q} < 0$, Table 4.1(b) shows us that we have a saddle point. By periodicity, $(\pm 3\pi, 0), (\pm 5\pi, 0), (\pm 7\pi, 0), \ldots$ are saddle points.

### Sec. 4.6 Nonhomogeneous Linear Systems of ODEs

In this section we return from nonlinear to linear systems of ODEs. The text explains that the transition from homogeneous to nonhomogeneous linear systems is quite similar to that for a single ODE. Namely, a general solution is the sum of a general solution $y^{(h)}$ of the homogeneous system plus a particular solution $y^{(p)}$ of the nonhomogeneous system, your main task is the determination of a $y^{(p)}$, either by undetermined coefficients or by variation of parameters. Undetermined coefficients is explained on p. 161. It is similar to that for single ODEs. The only difference is that in the Modification Rule you may need an extra term. For instance, if $e^{kt}$ appears in $y^{(h)}$, set $y^{(p)} = ut e^{kt} + ve^{kt}$ with the extra term $ve^{kt}$.

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3. **General solution.** $e^{3t}$ and $-3e^{3t}$ are such that we can apply the method of undetermined coefficients for determining a particular solution of the nonhomogeneous system. For this purpose we must first determine a general solution of the homogeneous system. The matrix of the latter is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

It has the characteristic equation $\lambda^2 - 1 = 0$. Hence the eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = 1$. Eigenvectors $x = x^{(1)}$ and $x^{(2)}$ are obtained from $(A - \lambda I)x = 0$ with $\lambda = \lambda_1 = -1$ and $\lambda = \lambda_2 = 1$, respectively. For $\lambda_1 = -1$ we obtain

$$x_1 + x_2 = 0, \quad \text{thus } x_2 = -x_1, \quad \text{say, } x_1 = 1, x_2 = -1.$$ 

Similarly, for $\lambda_2 = 1$ we obtain

$$-x_1 + x_2 = 0, \quad \text{thus } x_2 = x_1, \quad \text{say, } x_1 = 1, x_2 = 1.$$ 

Hence eigenvectors are $x^{(1)} = [1 \ -1]^T$ and $x^{(2)} = [1 \ 1]^T$. This gives the general solution of the homogeneous system

$$y^{(h)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-t}.$$ 

Now determine a particular solution of the nonhomogeneous system. Using the notation in the text (Sec. 4.6) we have on the right $g = [1 \ -3]^T e^{3t}$. This suggests the choice

(a) $$y^{(p)} = u e^{3t} = [u_1 \ u_2]^T e^{3t}.$$ 

Here $u$ is a constant vector to be determined. The Modification Rule is not needed because 3 is not an eigenvalue of $A$. Substitution of (a) into the given system $y' = Ay + g$ yields

$$y^{(p)' = 3ue^{3t} = Ay^{(p)} + g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}.$$
Omitting the common factor $e^{3t}$, you obtain, in terms of components,

\[3u_1 = u_2 + 1\]  
\[3u_2 = u_1 - 3\]

ordered

\[3u_1 - u_2 = 1,\]  
\[-u_1 + 3u_2 = -3.\]

Solution by elimination or by Cramer’s rule (Sec. 7.6) $u_1 = 0$ and $u_2 = -1$. Hence the answer is

\[y = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t}.\]

13. **Initial value problem.** The given system is

\[y_1' = y_2 - 5 \sin t,\]  
\[y_2' = -4y_1 + 17 \cos t,\]

where the initial conditions are $y_1(0) = 5, y_2(0) = 2$. First we have to solve the homogeneous system

\[y' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} y.\]

Its characteristic equation $\lambda^2 + 4 = 0$ has the roots $\pm 2i$. For $\lambda = 2i$ we obtain an eigenvector from

$-2ix_1 + x_2 = 0$, say, $x^{(1)} = [1 \quad 2i]^T$. For $\lambda = -2i$ we have $2ix_1 + x_2 = 0$, so that an eigenvector is, say, $x^{(2)} = [1 \quad -2i]^T$. You obtain the complex general solution of the homogeneous system as follows. We apply Euler’s formula twice.

\[y^{(h)} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}\]

\[= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} (\cos 2t + i \sin 2t) + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos(-2t) + i \sin(-2t))\]

\[= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} (\cos 2t + i \sin 2t) + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} (\cos 2t + i \sin 2t)\]

\[= \begin{bmatrix} c_1 \cos 2t + ic_1 \sin 2t + c_2 \cos 2t - ic_2 \sin 2t \\ 2ic_1 \cos 2t + 2it^2c_1 \sin 2t - 2ic_2 \cos 2t + 2it^2c_2 \sin 2t \end{bmatrix}\]

\[= \begin{bmatrix} c_1 \cos 2t + ic_1 \sin 2t + c_2 \cos 2t - ic_2 \sin 2t \\ 2ic_1 \cos 2t - 2c_1 \sin 2t - 2ic_2 \cos 2t - 2c_2 \sin 2t \end{bmatrix}\]

\[= \begin{bmatrix} (c_1 + c_2) \cos 2t + i(c_1 - c_2) \sin 2t \\ (2ic_1 - 2ic_2) \cos 2t + (-2c_1 - 2c_2) \sin 2t \end{bmatrix}\]

\[= \begin{bmatrix} c_1 + c_2 \\ 2ic_1 - 2ic_2 \end{bmatrix} \cos 2t + i \begin{bmatrix} c_1 - c_2 \\ 2ic_1 + 2ic_2 \end{bmatrix} \sin 2t\]

\[= \begin{bmatrix} A \\ B \end{bmatrix} \cos 2t + \begin{bmatrix} \frac{1}{2}B \\ -2A \end{bmatrix} \sin 2t\]
where

\[ A = c_1 + c_2, \quad B = 2i(c_1 - c_2). \]

We determine \( y^{(p)} \) by the method of undetermined coefficients, starting from

\[ y^{(p)} = u \cos t + v \sin t = \begin{bmatrix} u_1 \cot t + v_1 \sin t \\ u_2 \cos t + v_2 \sin t \end{bmatrix}. \]

Termwise differentiation yields

\[ y^{(p)_r} = \begin{bmatrix} -u_1 \sin t + v_1 \sin t \\ -u_2 \sin t + v_2 \cos t \end{bmatrix}. \]

In components

\[ y_1^{(p)_r} = -u_1 \sin t + v_1 \sin t, \quad y_2^{(p)_r} = -u_2 \sin t + v_2 \cos t. \]

Substituting this and its derivative into the given nonhomogeneous system, we obtain, in terms of components,

\[ -u_1 \sin t + v_1 \sin t = u_2 \cos t + v_2 \sin t - 5 \sin t, \]
\[ -u_2 \sin t + v_2 \cos t = -4u_1 \cos t - 4v_1 \sin t + 17 \cos t. \]

By equating the coefficients of the cosine and sine in the first of these two equations, we obtain

\[ (E1) \quad -u_1 = v_2 - 5, \quad (E2) \quad v_1 = u_2, \]
\[ (E3) \quad -u_2 = -4v_2, \quad (E4) \quad v_2 = -4u_1 + 17. \]

Substituting (E4) into (E1) \(-u_1 = -4u_1 + 17 - 5\) gives (E5) \( u_1 = \frac{12}{3} = 4).\]

Substituting (E5) into (E4) \( v_2 = -4(4) + 17 = 1 \) gives (E6) \( v_2 = 1 \).

(E2) and (E3) together \( u_2 = v_1 \) and \( u_2 = 4v_1 \) is only true for (E7) \( u_2 = v_1 = 0 \).

Equations (E5), (E6), (E7) form the solution to the homogeneous linear system, that is,

\[ u_1 = 4, \quad u_2 = 0, \quad v_1 = 0, \quad v_2 = 1. \]

This gives the general answer

\[ y_1 = y_1^{(h)} + y_1^{(p)} = A \cos 2t + \frac{i}{2}B \sin 2t + 4 \cos t, \]
\[ y_2 = y_2^{(h)} + y_2^{(p)} = B \cos 2t + \frac{i}{2}A \sin 2t + \sin t. \]

To solve the initial value problem, we use \( y_1(0) = 5, y_2(0) = 2 \) to obtain

\[ y_1(0) = A \cos 0 + \frac{i}{2}B \sin 0 + 4 \cos 0 = A \cdot 1 + 0 + 4 = 5 \quad \text{hence} \quad A = 1, \]
\[ y_2(0) = B \cos 0 - 2A \sin 0 + \sin 0 = B \cdot 1 - 0 + 0 = 2 \quad \text{hence} \quad B = 2. \]
Thus the final answer is
\[ y_1 = \cos 2t + \sin 2t + 4 \cos t, \]
\[ y_2 = 2 \cos 2t - 2 \sin 2t + \sin t. \]

17. **Network.** First derive the model. For the left loop of the electrical network you obtain, from Kirchhoff’s Voltage Law

(a) \[ LI_1' + R_1(I_1 - I_2) = E \]

because both currents flow through \( R_1 \), but in opposite directions, so that you have to take their difference. For the right loop you similarly obtain

(b) \[ R_1(I_2 - I_1) + R_2I_2 + \frac{1}{C} \int I_2 \, dt = 0. \]

Insert the given numerical values in (a). Do the same in (b) and differentiate (b) in order to get rid of the integral. This gives

\[ I_1' + 2(I_1 - I_2) = 200, \]
\[ 2(I_2' - I_1') + 8I_2' + 2I_2 = 0. \]

Write the terms in the first of these two equations in the usual order, obtaining

(a1) \[ I_1' = -2I_1 + 2I_2 + 200. \]

Do the same in the second equation as follows. Collecting terms and then dividing by 10, you first have

\[ 10I_2' - 2I_1' + 2I_2 = 0 \]

and then \[ I_2' - 0.2I_1' + 0.2I_2 = 0. \]

To obtain the usual form, you have to get rid of the term in \( I_1' \), which you replace by using (a1). This gives

\[ I_2' - 0.2(-2I_1 + 2I_2 + 200) + 0.2I_2 = 0. \]

Collecting terms and ordering them as usual, you obtain

(b1) \[ I_2' = -0.4I_1 + 0.2I_2 + 40. \]

(a1) and (b1) are the two equations of the system that you use in your further work. The matrix of the corresponding homogeneous system is

\[ A = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix}. \]

Its characteristic equation is (1 is the unit matrix)

\[ \det(A - \lambda I) = (-2 - \lambda)(0.2 - \lambda) - (-0.4) \cdot 2 = \lambda^2 + 1.8\lambda + 0.4 = 0. \]
This gives the eigenvalues

\[ \lambda_1 = -0.9 + \sqrt{0.41} = -0.259688 \]

and

\[ \lambda_2 = -0.9 - \sqrt{0.41} = -1.540312. \]

Eigenvectors are obtained from \((A - \lambda I)x = 0\) with \(\lambda = \lambda_1\) and \(\lambda = \lambda_2\). For \(\lambda_1\) this gives

\[ (-2 - \lambda_1)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_1. \]

Similarly, for \(\lambda_2\) you obtain

\[ (-2 - \lambda_2)x_1 + 2x_2 = 0, \quad \text{say,} \quad x_1 = 2 \quad \text{and} \quad x_2 = 2 + \lambda_2. \]

The eigenvectors thus obtained are

\[ x^{(1)} = \begin{bmatrix} 2 \\ 2 + \lambda_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 + \sqrt{0.41} \end{bmatrix}. \]

and

\[ x^{(2)} = \begin{bmatrix} 2 \\ 2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1.1 - \sqrt{0.41} \end{bmatrix}. \]

This gives as a general solution of the homogeneous system

\[ I^{(h)} = c_1 x^{(1)} e^{\lambda_1 t} + c_2 x^{(2)} e^{\lambda_2 t}. \]

You finally need a particular solution \(I^{(p)}\) of the given nonhomogeneous system \(J' = AJ + g\), where \(g = \begin{bmatrix} 200 \\ 40 \end{bmatrix}\) is constant, and \(J = [I_1 \quad I_2]^T\) is the vector of the currents. The method of undetermined coefficients applies. Since \(g\) is constant, you can choose a constant \(I^{(p)} = u = [u_1 \quad u_2]^T = \text{const}\) and substitute it into the system, obtaining, since \(u' = 0\),

\[ I^{(p)} = 0 = Au + g = \begin{bmatrix} -2 & 2 \\ -0.4 & 0.2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 200 \\ 40 \end{bmatrix} = \begin{bmatrix} -2u_1 + 2u_2 + 200 \\ -0.4u_1 + 0.2u_2 + 40 \end{bmatrix}. \]

Hence you can determine \(u_1\) and \(u_2\) from the system

\[ \begin{align*}
-2u_1 + 2u_2 &= -200, \\
-0.4u_1 + 0.2u_2 &= -40. 
\end{align*} \]

The solution is \(u_1 = 100, u_2 = 0\). The answer is

\[ J = I^{(h)} + I^{(p)}. \]