Chapter 1

Introduction

Operations research is an important field of applied mathematics with increasingly diverse uses in industry, agriculture, commerce, engineering, and government. It is expected that the benefits of these current uses, and of uses yet to be developed, will continue to grow as members of our society become more conscious of the limitations in our resources, and attach more importance to overall planning and to increases in productivity.

The historical roots of operations research can be traced back as far as the work of Daniel Bernoulli (1700–1782), who was concerned with how much a shipping merchant should be willing to pay for insurance on cargo. Bernoulli was intrigued by the question of why insurance should be worthwhile, even though the expected profit is less with insurance than without. It is only quite recently, during World War II, that operations research has come to be recognized as a field in its own right. The term "operations research" dates from this period, when quantitative methods were used by the British and Americans to handle the logistics of large-scale military "operations." These logistics problems involved the effective overseas supply of Great Britain during the Battle of Britain and later the supply and deployment of American forces in the Pacific.

Most of the methods and results that now are considered to constitute the field of operations research have been developed since that time. The range of applications has grown rapidly in business and in government, and the focus of attention has shifted to these areas. It has been suggested that "management science" would now be a more appropriate title for this field since it is more descriptive of current activity, but the term "operations research" seems to have the advantage of tradition.
It has never been easy to define what the domain of operations research is or should be. While this can be frustrating to one's sense of order, it can also be regarded as a sign of a young and vigorous field. Perhaps the single characteristic that best distinguishes operations research from other fields of applied mathematics is its emphasis on providing a person or group of persons with a method for making a decision that is optimal for their purposes, as opposed to providing them with a greater comprehension of the actual or predicted behavior of a situation being analyzed. It is true that other fields of applied mathematics, such as applied differential equations in engineering, often have aided a person in choosing an optimal course of action, but such decision making was arrived at informally and outside the actual analytic model. In contrast to this state of affairs, most models developed as part of an operations research investigation include a specific formulation of the decision maker's objectives as an essential part of the model. It is often assumed that he or she already has sufficient understanding and information concerning the situation at hand, and that the difficulty lies in how to make the best decision with this available information.

1.1. SOME LINEAR OPTIMIZATION MODELS

Speaking in general terms, one common source of difficulty in trying to decide upon an optimal course of action is the presence of a large number of different activities, all of which depend on the same limited resources. If the decision maker engages in one activity in a manner that is optimal for his goals, then insufficient resources will remain to support the other activities. How should he choose the best balance between the various activities that are competing for the available resources?

Example of a Furniture Manufacturer

To illustrate, imagine a furniture construction shop that specializes in two furniture items: a three-shelf bookcase and a study desk. The furniture produced is supplied at monthly intervals to a large retail chain. Accounting indicates that a net profit of $8 is made on each bookcase and a net profit of $12 is made on each desk produced. The retailing firm is able to accept any number of bookcases or desks likely to be built in the shop, and any items that are partially completed on one of the monthly pickup days can be finished during the next month.

The decision faced by the owner of the furniture shop is how to divide his limited facilities between the production of bookcases and the production of desks. Both items pass through two phases of production, assembly and finishing, which are done by separate groups of employees. Suppose that the information regarding labor requirements and availability is given in Table 1.1.

<table>
<thead>
<tr>
<th>Number of hours needed per bookcase</th>
<th>Number of hours needed per desk</th>
<th>Total number of hours available per month</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assembling</td>
<td>1.6</td>
<td>4.8</td>
</tr>
<tr>
<td>Finishing</td>
<td>2.5</td>
<td>3.0</td>
</tr>
</tbody>
</table>

Let $x_1$ denote the number of bookcases produced per month, and let $x_2$ denote the number of desks produced per month. We will represent a monthly production schedule by the two-dimensional vector $(x_1, x_2)$. The possible production schedules must satisfy several conditions. First, it is impossible to produce negative quantities of bookcases or desks, so we require that $x_1 \geq 0$ and $x_2 \geq 0$. Second, the total number of hours spent on the assembling phase of the bookshelf and desk construction must be less than or equal to the amount of labor available per month for this purpose. By Table 1.1 the number of hours spent assembling bookcases is $1.6x_1$, and the number of hours spent assembling desks is $4.8x_2$. Thus, the total number of hours spent per month on the assembling phase is $1.6x_1 + 4.8x_2$. Since the total number of hours available per month is 480, we require the linear inequality

$$1.6x_1 + 4.8x_2 \leq 480.$$ 

Similarly, the total number of hours spent on the finishing phase is $2.5x_1 + 3.0x_2$. This must be less than or equal to the maximum number of hours available per month for finishing, and so we require

$$2.5x_1 + 3.0x_2 \leq 450.$$

By combining these requirements, we see that the set of possible monthly production schedules $(x_1, x_2)$ corresponds to the polygonal region $S$ in Figure 1.1. The vertices have been calculated by solving the appropriate pairs of simultaneous equations. One might ask why we have formulated constraints on the amounts of labor available but have not formulated any constraints on the amounts of material available. The owner of the furniture shop felt that it would be able to purchase the amounts of materials needed for any monthly production schedule that would be possible, given his labor constraints. Considering the three possible limitations of insufficient materials, insufficient labor, and an insufficient market for his products, he felt that the limitations on labor were certain to be the "bottleneck" in his production plans.
The furniture store owner wishes to maximize his total net return per month. It was stated previously that each bookcase had a net return of $8 and each desk had a net return of $12; therefore, the total net return can be represented by the function

\[ z(x_1, x_2) = 8x_1 + 12x_2. \]

Thus, the goal is to maximize the function \( z(x_1, x_2) \).

The owner receives a gross return, that is, a price, of $45 per bookcase and $75 per desk. Why not maximize the gross return \( 45x_1 + 75x_2 \) rather than the net return \( 8x_1 + 12x_2 \)? The reason is this: the gross return does not include various expenses that depend on the variables \( x_1, x_2 \). First, there is the cost of labor, which is $5 per hour for each employee. It takes 4.1 hr to construct a bookcase, and 7.8 hr to construct a desk (as the reader should check); therefore, the labor costs are $20.50 per bookcase and $39.00 per desk. Second, there is the cost of materials, which is $16.50 per bookcase and $24.00 per desk. Taking these two expenses into account, the net returns are 45.00 - 20.50 - 16.50 = $8 per bookcase and 75.00 - 39.00 - 24.00 = $12 per desk. What about other expenses such as heating, the salary of a part-time office secretary, and depreciation on the facilities? These do not depend on the amounts \( x_1, x_2 \) of production. Therefore, they should be either included in the net return function \( z(x_1, x_2) \) as constant terms or not included at all. Such expenses should not be prorated, for example, by including an expense of $0.85 per bookcase or desk to cover office expenses.

The net return function \( z(x_1, x_2) \) defines a family of parallel lines as in Figure 1.2. Any two production schedules on the same line yield the same total net return, and higher lines correspond to greater returns. The maximum total net return is therefore the return associated with the highest line, which still intersects the region \( S \) of possible monthly production schedules.
The total net return per month can be calculated directly for each of the six vertices of the polyhedral region $S$. The largest of the resulting values is $1,500, corresponding to the vertex $(100, 66.7, 0)$. Thus, with a net return of $9 per unit on the end-table files, it would not increase the total net return to produce any of this item.

Suppose, however, that the net return per unit on the end-table files is $11 instead of $9. The same production schedules will be possible as before. However, a comparison of the net returns corresponding to each of the six vertices of $S$ now leads to a maximum value of $1,700, corresponding to the vertex $(0, 50, 100)$. Thus, with the larger net return of $11 on the third item, an optimal monthly production schedule is

$$x_1 = 0, \quad x_2 = 50, \quad x_3 = 100.$$ 

Note that no bookcases are to be produced. In a sense, the end-table files have replaced the bookcases as being more profitable.

It is intuitively plausible and will be proved later that for any formulation of the sort we have discussed, an optimal monthly production schedule will lie on a vertex of the region $S$ of all possible schedules. Moreover, the optimal production schedule will be unique unless the lines of equal return are parallel to an edge or a face of the region $S$. In this case, every point on the face or edge will represent an optimal schedule.

These observations lead us to a finite procedure for finding an optimal schedule. First, locate the vertices of the polygonal region $S$ of all possible schedules. Then, compute the net return for each vertex and compare these values.

This is indeed the procedure that we will use in this chapter. We will refer to it as the graphic method. It should be pointed out, however, that in practically all applications of the ideas we are discussing, there will be far more than two variables and four constraints. It is not unusual in practical problems for both the number of variables and the number of constraints to be in the thousands. Whereas the number of vertices will still be finite, and hence in theory their net returns could be directly compared, the number of vertices will be so immense that in practice a direct comparison is out of the question with even the largest of current computing facilities.

Fortunately, there are methods (algorithms) that avoid the necessity of considering all possible vertices. In Chapters 3 and 4 we will study such a method, the so-called simplex algorithm, discovered by George Dantzig in about 1947. Dantzig and his associates were trying to develop mathematical techniques for analyzing military programming and planning models for the U.S. Air Force. His models were therefore called linear programming models. The term “programming” connotes planning and has no direct connection with computer programming.

Example of a Dairy Farmer

Let us next consider an example of a linear programming model having a geometric diagram rather different from that of the furniture construction model. Consider a dairy farm in Minnesota with a large herd of Holstein cattle. During the winter months, these animals must be kept in and fed a combination of hay and oats. One of the farmer’s problems is to select a

<table>
<thead>
<tr>
<th>TABLE 1.2. Per unit contents and costs of hay and oats</th>
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<tbody>
<tr>
<td>Per unit of hay</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>Units of protein</td>
</tr>
<tr>
<td>Units of fat</td>
</tr>
<tr>
<td>Units of calcium</td>
</tr>
<tr>
<td>Units of phosphorus</td>
</tr>
<tr>
<td>Cost per unit</td>
</tr>
</tbody>
</table>
feed mixture at minimum cost that contains the nutrition needed to maintain the cows in a healthy and productive condition.

For the purpose of this discussion, suppose that the nutritional items of concern are digestible protein, fat, calcium, and phosphorus. The nutritive contents and the per unit costs of hay and oats are specified in Table 1.2 in terms of the units in which the farmer will be purchasing these foodstuffs.

Let \( x_1 \) denote the number of units of hay to be eaten per day by one of the Holstein cows, and let \( x_2 \) denote the analogous number of units of oats. Then any possible feed mixture \((x_1, x_2)\) of hay and oats must satisfy the linear inequalities:

\[
\begin{align*}
\text{protein:} & \quad 13.2x_1 + 34.0x_2 \geq 65.0 \\
\text{fat:} & \quad 4.3x_1 + 5.9x_2 \geq 14.0 \\
\text{calcium:} & \quad 0.02x_1 + 0.09x_2 \geq 0.12 \\
\text{phosphorus:} & \quad 0.04x_1 + 0.09x_2 \geq 0.15
\end{align*}
\]

\[x_1 \geq 0, \quad x_2 \geq 0.\]

The cost of any feed mixture will be

\[z = 0.66x_1 + 2.08x_2.\]

The set \( S \) of all possible feed mixtures is an unbounded subset of the plane. It is represented in Figure 1.4 as that part of the positive quadrant which lies in the intersection of the four half-planes defined by the nutrition requirements. Notice that the phosphorus requirement is super-

1.2. DEFINITION OF A LINEAR OPTIMIZATION MODEL

The mathematical definition of a linear optimization model can be stated immediately.

Definition 1.1. Let \( z \) denote a real valued function of \( n \) real variables \( x_1, x_2, \ldots, x_n \) of the form

\[z(x_1, x_2, \ldots, x_n) = r_1x_1 + r_2x_2 + \cdots + r_nx_n + d.\]

A linear optimization model consists of either the problem of maximizing the function \( z \) or the problem of minimizing the function \( z \) among those values of the variables \( x_1, x_2, \ldots, x_n \) that satisfy a given finite number of linear equalities and inequalities. In other words, \( x_1, x_2, \ldots, x_n \) must satisfy a list of constraints, each having one of the forms:

\[
\begin{align*}
a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \\
a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b, \\
a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b.
\end{align*}
\]

To illustrate, a linear optimization model might consist of the problem of finding values for the variables \( x_1, x_2, x_3, x_4 \) that maximize the function

\[z = 3.7x_1 + 2.5x_2 + 1.9x_3 - 0.8x_4 - 12.0\]

subject to the linear equalities and inequalities:

\[
\begin{align*}
2.2x_1 + 2.0x_2 + 1.5x_3 - x_4 &= 9.6 \\
x_1 + 3.2x_2 + x_3 + 4.6x_4 &\leq 7.9 \\
x_1 + x_3 &= 6.0 \\
x_2 &= x_4 \geq 0. \\
x_3 \geq 0, \quad x_4 \geq 0.
\end{align*}
\]
In the analysis of a linear optimization model, inequalities of the form \( x_j \geq 0 \) are treated differently from the other constraints in the model. Any variable \( x_j \) that is constrained by requiring \( x_j \geq 0 \) is called a nonnegative variable. Any variable that is not so constrained is called a free variable. Thus, in the above example, the variables \( x_1 \) and \( x_2 \) are nonnegative variables, and the variables \( x_3 \) and \( x_4 \) are free variables.

In general, the function \( z \) denotes that quantity, such as net return, which the decision maker wishes to make as large (or as small) as possible. The function refers to the goal or objective involved in the situation. For this reason, \( z \) is called the objective function. The variables \( x_1, x_2, \ldots, x_n \) denote those quantities, such as the amounts of various goods to be manufactured, the values of which the decision maker is able to control or decide upon. Thus, these variables are called the decision variables.

A certain amount of terminology has come into being during the development of operations research that is peculiar to the field and is often equivalent to mathematical terminology of more general usage. For example, the terms linear optimization and linear programming are equivalent. The function \( z \) can be referred to either more mathematically as the optimizing function or more traditionally as the objective function. In such a situation, it seems advisable to learn both terms and then mentally draw an equality sign between them.

**Definition 1.2** Any values of the variables \( x_1, \ldots, x_n \), such that all the equalities and inequalities of a linear optimization model are satisfied (including any nonnegativity conditions) are called a solution of the model. Any such values are also called a feasible solution. The set of all solutions is called the set of feasible solutions or the feasible region. Any solution \( x_1, \ldots, x_n \) that optimizes the objective function among the set of all solutions is called an optimal solution. The resulting value of the objective function is called the optimal value of the model or more briefly the optimal value.

For example, the linear optimization model for the furniture manufacturer who produces bookshelves and desks has as its feasible solutions the set \( S \) shown in Figure 1.1, and as its only optimal solution the values \( x_1 = 100 \), \( x_2 = 66.7 \). The optimal value of the model is \( z = 1,600 \).

1.3. A LOOK AHEAD

We have seen that one source of a decision maker's difficulties may be the sheer number and variety of alternative decisions that must be considered. This can be an enormous problem even when the situation is favorable in the following way; the decision maker has all the relevant information and has complete control of the situation in that the consequences of the possible decisions are determined and known. For example, the dairy farmer knew what the cost of hay and oats would be as a consequence of any purchasing decision. A mathematical model including this kind of assumption as part of its formulation is referred to as a deterministic model. The linear programming models developed in Section 1.1 are examples of deterministic models.

A conceptually different source of difficulty for a decision maker arises when the assumption of complete information and control is not appropriate to the situation. A mathematical model including a formulation of this source of difficulty is often referred to as a probabilistic model or a model involving uncertainty.

For the next several chapters, we will be concerned with the difficulties that arise in the use of deterministic models, where the decision maker is confronted with a large and complicated class of alternative decisions. This will lead us to a study of the formulation and mathematical analysis of linear optimization models, and then to a study of several other closely related optimization models.

Following this material, we will consider the difficulties facing a decision maker who must decide upon a course of action in a situation involving uncertainty. This will lead us to study models involving probabilities as the quantification of this uncertainty, and then to study systematic procedures for quantifying a decision maker's objectives in situations involving uncertainty.

While this material will not provide you with an introduction to all of the important areas and techniques of operations research, it will provide you with a good understanding of the basic methodology of this increasingly important field of applied mathematics. Bon voyage!

**PROBLEMS**

For each of the linear optimization models of Problems 1–6, draw a diagram of the set of feasible solutions. Use the graphic method to find the optimal solution and the optimal value of the model.

1. \[ \text{maximize} \quad z = 5x_1 + 4x_2 \]
   \[ \text{subject to} \quad x_1 + 2x_2 \leq 14 \]
   \[ 3x_1 + 2x_2 \leq 24 \]
   \[ x_1 + x_2 \leq 9 \]
   \[ x_1 \geq 0, \quad x_2 \geq 0. \]

2. \[ \text{minimize} \quad z = 2x_1 + 3x_2 \]
   \[ \text{subject to} \quad x_1 + 4x_2 \geq 16 \]
   \[ x_1 + 2x_2 \geq 12 \]
   \[ x_1 + x_2 \geq 8 \]
   \[ x_1 \geq 0, \quad x_2 \geq 0. \]
11. Gretchen, a graduate student, has decided to eat only granola (a natural cereal) during the coming academic year in order to save money. She must decide how much granola to eat and how much whole milk to pour on the cereal in order to obtain the necessary nutrition at lowest cost. Gretchen will supplement her diet with whatever vitamins and minerals are necessary, and so the only nutritional factors she will consider are calories, protein, and calcium. Assuming the information in Table 1.3, how much granola and milk should Gretchen eat each day in order to meet the daily allowances at lowest cost? What will be the lowest cost be?

<table>
<thead>
<tr>
<th>TABLE 1.3. Nutritional contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Per oz of granola</td>
</tr>
<tr>
<td>-------------------</td>
</tr>
<tr>
<td>Calories</td>
</tr>
<tr>
<td>Protein (gm)</td>
</tr>
<tr>
<td>Calcium (gm)</td>
</tr>
<tr>
<td>Cost per unit</td>
</tr>
</tbody>
</table>

12. A manufacturing organization produces a variety of toys somewhere up near the North Pole. This year the outfit is particularly concerned with the production of two types of metal toys: tricycles and wagons. The three most important materials used to produce these toys are medium-weight sheet metal, tubular steel, and solid rubber tires. There are on hand 48 units of sheet metal, 48 units of tubular steel, 48 small tires (to be used for the rear wheels on the tricycles and all four wheels on the wagons), and 12 large tires (to be used for the front wheels on the tricycles). If the amount of each material needed per tricycle and per wagon is as in Table 1.4, how many tricycles and wagons should be produced by December 25 in order that the sum of tricycles and wagons produced might be as large as possible?

<table>
<thead>
<tr>
<th>TABLE 1.4. Number of units required per toy produced</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>Sheet metal</td>
</tr>
<tr>
<td>Tubular steel</td>
</tr>
<tr>
<td>Small tires</td>
</tr>
<tr>
<td>Large tires</td>
</tr>
</tbody>
</table>

3. Maximize
   \[ z = 2x_1 + 5x_2 \]
   subject to
   \[ \begin{align*}
   10x_1 + 7x_2 & \leq 70 \\
   8x_1 + 10x_2 & \leq 80 \\
   x_2 & \leq 5 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0. \\
   \end{align*} \]

4. Minimize
   \[ z = 2x_1 + 5x_3 \]
   subject to
   \[ \begin{align*}
   10x_1 + 7x_3 & \geq 70 \\
   8x_1 + 10x_3 & \geq 80 \\
   x_2 & \geq 5 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0. \\
   \end{align*} \]

5. Maximize
   \[ z = 16x_1 + 12x_2 + 8x_3 \]
   subject to
   \[ \begin{align*}
   x_1 + x_2 + x_3 & \leq 2 \\
   4x_1 + 2x_2 + 3x_3 & \leq 3 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0. \\
   \end{align*} \]

6. Maximize
   \[ z = x_1 + x_2 + x_3 \]
   subject to
   \[ \begin{align*}
   3x_1 + 2x_2 + 4x_3 & \leq 12 \\
   2x_1 + 3x_2 + 2x_3 & \leq 12 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0 \\
   x_3 & \geq 0. \\
   \end{align*} \]

7. Consider the set \( S \) of values \( x_1, x_2 \) satisfying
   \[ \begin{align*}
   20 & \leq 4x_1 + 5x_2 \leq 60 \\
   12 & \leq 4x_1 + x_2 \leq 20. \\
   \end{align*} \]

   Find the maximum and minimum values of the objective function \( z = -2x_1 + 5x_2 \) over the set \( S \).

8. Consider the set \( S \) of values \( x_1, x_2 \) satisfying
   \[ \begin{align*}
   2x_1 + 7x_2 & \geq 28 \\
   3x_1 + 5x_2 & \geq 30 \\
   2x_1 + x_2 & \geq 12 \\
   x_1 & \geq 0 \\
   x_2 & \geq 0. \\
   \end{align*} \]

   Show that the objective function \( z = x_1 - 4x_2 \) has no minimum value over the set \( S \).

9. Show that the function \( z = 6x_1 + 10x_2 \) has its minimum value over the set \( S \) of Problem 8 at more than one solution \( x_1, x_2 \). Find all such optimal solutions.

10. Old MacDonald had a farm with 160 acres of pasture land. And on this farm he had some beef cattle, with each steer needing about one acre of land to graze. And on this farm he had some horses, with each horse needing about 1.5 acres of land to graze.

    In early spring, MacDonald can purchase beef cattle at $40 per calf, and horses at $80 per colt. In the fall, he can sell the grown beef cattle for $250 apiece, and the horses for $400 apiece. If one spring, he had $7,200 capital for purchasing calves and colts so that when he sells the grown cattle and horses in the following fall, he will maximize his net return (income from the sale in the fall minus purchase costs in the spring)?